On further differentiating w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{3}{2} \cdot \frac{d}{dt} t \cdot \frac{dt}{dx}$$

$$= \frac{3}{2} \cdot \frac{1}{2t} \qquad \left[\because \frac{dt}{dx} = \frac{1}{2t}\right]$$

$$= \frac{3}{4t}$$

**Q. 95** The value of $c$ in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(a) 1 (b) $-1$ (c) $\frac{3}{2}$ (d) $\frac{1}{3}$

**Sol.** *(a)* $\because$ $f'(c) = 0$ $\quad [\because f'(x) = 3x^2 - 3]$

$\Rightarrow$ $3c^2 - 3 = 0$

$\Rightarrow$ $c^2 = \frac{3}{3} = 1$

$\Rightarrow$ $c = \pm 1$, where $1 \in (0, \sqrt{3})$

$\therefore$ $c = 1$

**Q. 96** For the function $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$, the value of $c$ for mean value theorem is

(a) 1 (b) $\sqrt{3}$

(c) 2 (d) None of these

**Sol.** *(b)* $\because$ $f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow$ $1 - \frac{1}{c^2} = \frac{\left[3 + \frac{1}{3}\right] - \left[1 + \frac{1}{1}\right]}{3 - 1}$ $\quad \left[\because f'(x) = 1 - \frac{1}{x^2} \text{ and } b = 3, a = 1\right]$

$\Rightarrow$ $\frac{c^2 - 1}{c^2} = \frac{\frac{10}{3} - 2}{2}$

$\Rightarrow$ $\frac{c^2 - 1}{c^2} = \frac{4}{3 \times 2} = \frac{2}{3}$

$\Rightarrow$ $3(c^2 - 1) = 2c^2$

$\Rightarrow$ $3c^2 - 2c^2 = 3$

$\Rightarrow$ $c^2 = 3 \Rightarrow c = \pm \sqrt{3}$

$\because$ $c = \sqrt{3} \in (1, 3)$

**Q. 92** If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\cos x}{2y - 1}$ (b) $\frac{\cos x}{1 - 2y}$ (c) $\frac{\sin x}{1 - 2y}$ (d) $\frac{\sin x}{2y - 1}$

**Sol.** *(a)* $\because$ $y = (\sin x + y)^{1/2}$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{2}(\sin x + y)^{-1/2} \cdot \frac{d}{dx}(\sin x + y)$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{(\sin x + y)^{1/2}} \cdot \left(\cos x + \frac{dy}{dx}\right)$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{1}{2y}\left(\cos x + \frac{dy}{dx}\right) \quad [\because (\sin x + y)^{1/2} = y]$$

$\Rightarrow$ $$\frac{dy}{dx}\left(1 - \frac{1}{2y}\right) = \frac{\cos x}{2y}$$

$\therefore$ $$\frac{dy}{dx} = \frac{\cos x}{2y} \cdot \frac{2y}{2y - 1} = \frac{\cos x}{2y - 1}$$

**Q. 93** The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\cos^{-1} x$ is

(a) 2 (b) $\frac{-1}{2\sqrt{1 - x^2}}$

(b) $\frac{2}{x}$ (d) $1 - x^2$

**Sol.** *(a)* Let $u = \cos^{-1}(2x^2 - 1)$ and $v = \cos^{-1} x$

$\therefore$ $$\frac{dv}{dx} = \frac{+ - 1}{\sqrt{1 - (2x^2 - 1)^2}} \cdot 4x = \frac{- 4x}{\sqrt{1 - (4x^4 + 1 - 4x^2)}}$$

$$= \frac{- 4x}{\sqrt{- 4x^4 + 4x^2}} = \frac{- 4x}{\sqrt{4x^2 (1 - x^2)}}$$

$$= \frac{-2}{\sqrt{1 - x^2}}$$

and $$\frac{du}{dx} = \frac{- 1}{\sqrt{1 - x^2}}$$

$\therefore$ $$\frac{dx}{dv} = \frac{du / dx}{dv / dx} = \frac{- 2 / \sqrt{1 - x^2}}{-1 / \sqrt{1 - x^2}} = 2$$

**Q. 94** If $x = t^2$ and $y = t^3$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{4t}$ (c) $\frac{3}{2t}$ (d) $\frac{3}{2t}$

**Sol.** *(b)* We have, $x = t^2$ and $y = t^3$

$\therefore$ $$\frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 3t^2$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy / dt}{dx / dt} = \frac{3t^2}{2t} = \frac{3}{2}t$$

$\therefore$ LHL $= \lim_{x \to \frac{\pi^-}{2}} (mx + 1) = \lim_{h \to 0} \left[ m\left(\frac{\pi}{2} - h\right) + 1 \right] = \frac{m\pi}{2} + 1$

and RHL $= \lim_{x \to \frac{\pi^+}{2}} (\sin x + n) = \lim_{h \to 0} \left[ \sin\left(\frac{\pi}{2} + h\right) + n \right]$

$= \lim_{h \to 0} \cos h + n = 1 + n$

$\therefore$ LHL = RHL $\left[\text{to be continuous at } x = \frac{\pi}{2}\right]$

$\Rightarrow \quad m \cdot \frac{\pi}{2} + 1 = n + 1$

$\therefore \quad n = m \cdot \frac{\pi}{2}$

**Q. 90** If $f(x) = |\sin x|$, then

(a) $f$ is everywhere differentiable

(b) $f$ is everywhere continuous but not differentiable at $x = n\pi, n \in Z$

(c) $f$ is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}, n \in Z$

(d) None of the above

**Sol.** *(b)* We have, $f(x) = |\sin x|$

Let $f(x) = vou\ (x) = v\ [u(x)]$ [where, $u\ (x) = \sin x$ and $v\ (x) = |x|$]

$= v\ (\sin x) = |\sin x|$

where, $u(x)$ and $v\ (x)$ are both continuous.

Hence, $f(x) = vo\ u(x)$ is also a continuous function but $v(x)$ is not differentiable at $x = 0$.

So, $f(x)$ is not differentiable where $\sin x = 0 \Rightarrow x = n\pi,\ n \in Z$

Hence, $f(x)$ is continuous everywhere but not differentiable at $x = n\pi,\ n \in Z$.

**Q. 91** If $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{4x^3}{1 - x^4}$ (b) $\frac{-4x}{1 - x^4}$ (c) $\frac{1}{4 - x^4}$ (d) $\frac{-4x^3}{1 - x^4}$

**Sol.** *(b)* We have, $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{1 - x^2}{1 + x^2}} \cdot \frac{d}{dx}\left(\frac{1 - x^2}{1 + x^2}\right)$$

$$= \frac{(1 + x^2)}{(1 - x^2)} \cdot \frac{(1 + x^2) \cdot (-2x) - (1 - x^2) \cdot 2x}{(1 + x^2)^2}$$

$$= \frac{-2x[1 + x^2 + 1 - x^2]}{(1 - x^2) \cdot (1 + x^2)} = \frac{-4x}{1 - x^4}$$

**Q. 86** The function $f(x) = \cot x$ is discontinuous on the set

(a) $\{x = n\pi : n \in Z\}$
(b) $\{x = 2n\pi : n \in Z\}$
(c) $\left\{ x = (2n+1)\frac{\pi}{2}; n \in Z \right\}$
(d) $\left\{ x = \frac{n\pi}{2}; n \in Z \right\}$

**Sol.** *(a)* We know that, $f(x) = \cot x$ is continuous in $R - \{n\pi : n \in Z\}$.

Since, $f(x) = \cot x = \frac{\cos x}{\sin x}$ [since, $\sin x = 0$ at $n\pi$, $n \in Z$]

Hence, $f(x) = \cot x$ is discontinuous on the set $\{x = n\pi : n \in Z\}$.

**Q. 87** The function $f(x) = e^{|x|}$ is

(a) continuous everywhere but not differentiable at $x = 0$
(b) continuous and differentiable everywhere
(c) not continuous at $x = 0$
(d) None of the above

**Sol.** *(a)* Let $u(x) = |x|$ and $v(x) = e^x$

$$\therefore \quad f(x) = vou(x) = v[u(x)] = v|x| = e^{|x|}$$

Since, $u(x)$ and $v(x)$ are both continuous functions.

So, $f(x)$ is also continuous function but $u(x) = |x|$ is not differentiable at $x = 0$, whereas $v(x) = e^x$ is differentiable at everywhere.

Hence, $f(x)$ is continuous everywhere but not differentiable at $x = 0$.

**Q. 88** If $f(x) = x^2 \sin \frac{1}{x}$, where $x \neq 0$, then the value of the function $f$ at $x = 0$, so that the function is continuous at $x = 0$, is

(a) 0
(b) $-1$
(c) 1
(d) None of these

**Sol.** *(a)* $\because f(x) = x^2 \sin\left(\frac{1}{x}\right)$, where $x \neq 0$

Hence, value of the function $f$ at $x = 0$, so that it is continuous at $x = 0$ is 0.

**Q. 89** If $f(x) = \begin{cases} mx + 1, & \text{if } x \leq \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then

(a) $m = 1, n = 0$
(b) $m = \frac{n\pi}{2} + 1$
(c) $n = \frac{m\pi}{2}$
(d) $m = n = \frac{\pi}{2}$

**Sol.** *(c)* We have, $f(x) = \begin{cases} mx + 1, & \text{if } x \leq \frac{\pi}{2} \\ (\sin x + n), & \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$

**Q. 84** The function $f(x) = \dfrac{4 - x^2}{4x - x^3}$ is

(a) discontinuous at only one point
(b) discontinuous at exactly two points
(c) discontinuous at exactly three points
(d) None of the above

**Sol.** *(c)* We have,

$$f(x) = \frac{4 - x^2}{4x - x^3} = \frac{(4 - x^2)}{x(4 - x^2)}$$

$$= \frac{(4 - x^2)}{x(2^2 - x^2)} = \frac{4 - x^2}{x(2 + x)(2 - x)}$$

Clearly, $f(x)$ is discontinuous at exactly three points $x = 0$, $x = -2$ and $x = 2$.

**Q. 85** The set of points where the function $f$ given by $f(x) = |2x - 1| \sin x$ is differentiable is

(a) $R$ (b) $R - \left(\frac{1}{2}\right)$

(c) $(0, \infty)$ (d) None of these

**Sol.** *(b)* We have, $f(x) = |2x - 1| \sin x$

At $x = \frac{1}{2}$, $f(x)$ is not differentiable.

Hence, $f(x)$ is differentiable in $R - \left(\frac{1}{2}\right)$.

$\because$
$$Rf'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2} + h\right) - 1\right| \sin\left(\frac{1}{2} + h\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{|2h| \cdot \sin\left(\frac{1 + 2h}{2}\right)}{h} = 2 \cdot \sin\frac{1}{2}$$

and
$$Lf'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2} - h\right) - f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2} - h\right)^{-1}\right| - \sin\left(\frac{1}{2} - h\right) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{|0 - 2h| - \sin\left(\frac{1}{2} - h\right)}{-h} = -2\sin\left(\frac{1}{2}\right)$$

$\because$
$$Rf'\left(\frac{1}{2}\right) \neq Lf'\left(\frac{1}{2}\right)$$

So, $f(x)$ is not differentiable at $x = \frac{1}{2}$.

On, differentiating Eq. (ii) w.r.t. $x$, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\tan x\cdot\frac{1}{x}+\log x\cdot\sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx}=u\left[\frac{\tan x}{x}+\log x\cdot\sec^2 x\right]$$

$$=x^{\tan x}\left[\frac{\tan x}{x}+\log x\cdot\sec^2 x\right] \quad \text{...(iv)}$$

Also, differentiating Eq. (iii) w.r.t. $x$, we get

$$2v\cdot\frac{dv}{dx}=\frac{1}{2}(2x)\Rightarrow\frac{dv}{dx}=\frac{1}{4v}\cdot(2x)$$

$$\Rightarrow \quad \frac{dv}{dx}=\frac{1}{4\cdot\sqrt{\dfrac{x^2+1}{2}}}\cdot 2x=\frac{x\cdot\sqrt{2}}{2\sqrt{x^2+1}}$$

$$\Rightarrow \quad \frac{dv}{dx}=\frac{x}{\sqrt{2(x^2+1)}} \quad \text{...(v)}$$

Now, $\quad y=u+v$

$$\therefore \quad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$=x^{\tan x}\left[\frac{\tan x}{x}+\log x\cdot\sec^2 x\right]+\frac{x}{\sqrt{2(x^2+1)}}$$

# Objective Type Questions

**Q. 83** If $f(x)=2x$ and $g(x)=\dfrac{x^2}{2}+1$, then which of the following can be a discontinuous function?

(a) $f(x)+g(x)$ (b) $f(x)-g(x)$

(c) $f(x)\cdot g(x)$ (d) $\dfrac{g(x)}{f(x)}$

**Sol.** *(d)* We know that, if $f$ and $g$ be continuous functions, then

**(a)** $f+g$ is continuous **(b)** $f-g$ is continuous.

**(c)** $fg$ is continuous **(d)** $\dfrac{f}{g}$ is continuous at these points, where $g(x)\neq 0$.

Here, $$\frac{g(x)}{f(x)}=\frac{\dfrac{x^2}{2}+1}{2x}=\frac{x^2+2}{4x}$$

which is discontinuous at $x=0$.

**Q. 81** If $x = \sin t$ and $y = \sin pt$, then prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0.$$

**Sol.** We have, $x = \sin t$ and $y = \sin pt$

$$\therefore \quad \frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = \cos pt \cdot p$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p \cdot \cos pt}{\cos t} \qquad \ldots(i)$$

Again, differentiating both sides w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{\cos t \cdot \frac{d}{dt}(p \cdot \cos pt)\frac{dt}{dx} - p\cos pt \cdot \frac{d}{dt}\cos t \cdot \frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[\cos t \cdot p \cdot (-\sin pt) \cdot p - p\cos pt \cdot (-\sin t)]\frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[-p^2 \sin pt \cdot \cos t + p\sin t \cdot \cos pt] \cdot \frac{1}{\cos t}}{\cos^2 t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t}{\cos^3 t} \qquad \ldots(ii)$$

Since, we have to prove

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

$$\therefore \quad \text{LHS} = (1 - \sin^2 t)\frac{[-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t]}{\cos^3 t} - \sin t \cdot \frac{p\cos pt}{\cos t} + p^2 \sin pt$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix} (1 - \sin^2 t)(-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t) \\ - p\cos pt \cdot \sin t \cdot \cos^2 t + p^2 \sin pt \cdot \cos^3 t \end{bmatrix}$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix} -p^2 \sin pt \cdot \cos^3 t + p\cos pt \cdot \sin t \cdot \cos^2 t \\ - p\cos pt \cdot \sin t \cdot \cos^2 t + p^2 \sin pt \cdot \cos^3 t \end{bmatrix} \quad [\because 1 - \sin^2 t = \cos^2 t]$$

$$= \frac{1}{\cos^3 t} \cdot 0$$

$$= 0$$

**Hence proved.**

**Q. 82** Find the value of $\frac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\frac{x^2 + 1}{2}}$.

**Sol.** We have,

$$y = x^{\tan x} + \sqrt{\frac{x^2 + 1}{2}} \qquad \ldots(i)$$

Taking $u = x^{\tan x}$ and $v = \sqrt{\frac{x^2 + 1}{2}}$,

$$\log u = \tan x \log x \qquad \ldots(ii)$$

and

$$v^2 = \frac{x^2 + 1}{2} \qquad \ldots(iii)$$

**Q. 80** If $x^m \cdot y^n = (x+y)^{m+n}$, prove that

(i) $\dfrac{dy}{dx} = \dfrac{y}{x}$ and (ii) $\dfrac{d^2y}{dx^2} = 0$

**Sol.** We have, $$x^m \cdot y^n = (x+y)^{m+n} \quad \ldots(i)$$

**(i)** Differentiating Eq. (i) w.r.t. $x$, we get

$$\frac{d}{dx}(x^m \cdot y^n) = \frac{d}{dx}(x+y)^{m+n}$$

$$\Rightarrow \quad x^m \cdot \frac{d}{dy} y^n \cdot \frac{dy}{dx} + y^n \cdot \frac{d}{dx} x^m = (m+n)(x+y)^{m+n-1} \frac{d}{dx}(x+y)$$

$$\Rightarrow \quad x^m \cdot ny^{n-1} \frac{dy}{dx} + y^n \cdot mx^{m-1} = (m+n)(x+y)^{m+n-1}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}[x^m \cdot ny^{n-1} - (m+n)\cdot(x+y)^{m+n-1}] = (m+n)(x+y)^{m+n-1} - y^n mx^{m-1}$$

$$\Rightarrow \quad \frac{dy}{dx}[nx^m y^{n-1} - (m+n)(x+y)^{m+n-1}] = (m+n)\cdot(x+y)^{m+n-1} - \frac{y^{n-1}\cdot y \cdot mx^m}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{(m+n)(x+y)^{m+n}}{(x+y)} - \dfrac{y^{n-1}\cdot y \cdot mx^m}{x}}{\dfrac{n\,x^m y^n}{y} - (m+n)(x+y)^{m+n}\dfrac{1}{(x+y)}}$$

$$= \frac{\dfrac{x(m+n)(x+y)^{m+n} - (x+y)\cdot y^{n-1}\, y \cdot mx^m}{(x+y)\cdot x}}{\dfrac{(x+y)\,n\,x^m y^n - y(m+n)(x+y)^{m+n}}{(x+y)\cdot y}}$$

$$= \frac{\dfrac{x(m+n)\cdot x^m \cdot y^n - m(x+y)\,y^n\,x^m}{(x+y)\cdot x}}{\dfrac{(x+y)\,nx^m \cdot y^n - y(m+n)\cdot x^m \cdot y^n}{(x+y)\cdot y}} \quad [\because (x+y)^{m+n} = x^m \cdot y^n]$$

$$= \frac{x^m\, y^n [mx + nx - mx - my]\cdot(x+y)\,y}{x^m\, y^n[nx + ny - my - ny]\cdot(x+y)\cdot x}$$

$$= \frac{y}{x} \quad \ldots(ii)$$

**Hence proved.**

**(ii)** Further, differentiating Eq. (ii) *i.e.*, $\dfrac{dy}{dx} = \dfrac{y}{x}$ on both the sides w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{x\cdot\dfrac{dy}{dx} - y\cdot 1}{x^2}$$

$$= \frac{x\cdot\dfrac{y}{x} - y}{x^2} \quad \left[\because \frac{dy}{dx} = \frac{y}{x}\right]$$

$$= 0$$

**Hence proved.**

## Long Answer Type Questions

**Q. 79** Find the values of $p$ and $q$, so that $f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \le 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

**Sol.** We have, $f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \le 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

$\therefore \quad Lf'(1) = \lim_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1}$

$= \lim_{x \to 1^-} \dfrac{(x^2 + 3x + p) - (1 + 3 + p)}{x - 1}$

$= \lim_{h \to 0} \dfrac{[(1-h)^2 + 3(1-h) + p] - [1 + 3 + p]}{(1-h) - 1}$

$= \lim_{h \to 0} \dfrac{[1 + h^2 - 2h + 3 - 3h + p] - [4 + p]}{-h}$

$= \lim_{h \to 0} \dfrac{[h^2 - 5h + p + 4 - 4 - p]}{-h} = \lim_{h \to 0} \dfrac{h[h-5]}{-h}$

$= \lim_{h \to 0} -[h - 5] = 5$

$Rf'(1) = \lim_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \dfrac{(qx + 2) - (1 + 3 + p)}{x - 1}$

$= \lim_{h \to 0} \dfrac{[q(1+h) + 2] - (4 + p)}{1 + h - 1}$

$= \lim_{h \to 0} \dfrac{[q + qh + 2 - 4 - p]}{h} = \lim_{h \to 0} \dfrac{qh + (q - 2 - p)}{h}$

$\Rightarrow \quad q - 2 - p = 0 \Rightarrow p - q = -2$ ...(i)

$\Rightarrow \quad \lim_{h \to 0} \dfrac{qh + 0}{h} = q$ [for existing the limit]

If $Lf'(1) = Rf'(1)$, then $5 = q$

$\Rightarrow \quad p - 5 = -2 \Rightarrow p = 3$

$\therefore \quad p = 3$ and $q = 5$

$$\Rightarrow \quad x = -\sin a \cdot \cot(a+y)$$

$$\therefore \quad \frac{dx}{dy} = -\sin a \cdot [-\operatorname{cosec}^2(a+y)] \cdot \frac{d}{dy}(a+y)$$

$$= \sin a \cdot \frac{1}{\sin^2(a+y)} \cdot 1$$

$$= \frac{\sin^2(a+y)}{\sin a}$$

**Hence proved.**

## Q. 63 If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

**Sol.** We have,

$$\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$$

On putting $x = \sin\alpha$ and $y = \sin\beta$, we get

$$\sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$$

$$\Rightarrow \quad \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$$

$$\Rightarrow \quad 2\cos\frac{\alpha+\beta}{2}.\cos\frac{\alpha-\beta}{2} = a\left(2\cos\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2} = a\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \quad \cot\frac{\alpha-\beta}{2} = a$$

$$\Rightarrow \quad \frac{\alpha-\beta}{2} = \cot^{-1}a$$

$$\Rightarrow \quad \alpha - \beta = 2\cot^{-1}a$$

$$\Rightarrow \quad \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a \qquad [\because x = \sin\alpha \text{ and } y = \sin\beta]$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}$$

**Hence proved.**

## Q. 64 If $y = \tan^{-1}x$, then find $\frac{d^2y}{dx^2}$ in terms of $y$ alone.

**Sol.** We have,

$$y = \tan^{-1}x \qquad \text{[on differentiating w.r.t. } x\text{]}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2} \qquad \text{[again differentiating w.r.t. } x\text{]}$$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1+x^2)^{-1}$$

$$= -1(1+x^2)^{-2}.\frac{d}{dx}(1+x^2)$$

$$= -\frac{1}{(1+x^2)^2}\cdot 2x$$

$$= \frac{-2\tan y}{(1+\tan^2 y)^2} \qquad [\because y = \tan^{-1}x \Rightarrow \tan y = x]$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{x\cdot\frac{d}{dx}(y-x)-(y-x)\cdot\frac{d}{dx}\cdot x}{x^2}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{x\left(\frac{dy}{dx}-1\right)-(y-x)}{x^2}$$

$$\Rightarrow \quad \frac{x^2}{y}\cdot\frac{dy}{dx} = x\frac{dy}{dx} - x - y + x$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{x^2}{y}-x\right) = -y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2}{x^2-xy} = \frac{-y^2}{x(x-y)}$$

$$= \frac{y^2}{x(y-x)}\cdot\frac{x}{x} = \frac{y^2}{x^2}\cdot\frac{1}{\frac{(y-x)}{x}}$$

$$= \frac{(1+\log y)^2}{\log y} \quad \left[\because \log y = \frac{y-x}{x} \quad \log y = \frac{y}{x} - 1 \Rightarrow 1+\log y = \frac{y}{x}\right]$$

**Hence proved.**

**Q. 61** If $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$, then show that $\dfrac{dy}{dx} = \dfrac{y^2 \tan x}{y \log \cos x - 1}$.

**Sol.** We have,

$$y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$$

$$\Rightarrow \quad y = (\cos x)^y$$

$$\therefore \quad \log y = \log(\cos x)^y$$

$$\Rightarrow \quad \log y = y \log \cos x$$

On differentiating w.r.t. $x$, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{d}{dx}\log\cos x + \log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{\cos x}\cdot\frac{d}{dx}\cos x + \log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y} - \log\cos x\right] = \frac{-y\sin x}{\cos x} = -y\tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2\tan x}{(1-y\log\cos x)}$$

$$= \frac{y^2\tan x}{y\log\cos x - 1}$$

**Hence proved.**

**Q. 62** If $x\sin(a+y) + \sin a\cdot\cos(a+y) = 0$, then prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

**Sol.** We have,

$$x\sin(a+y) + \sin a\cdot\cos(a+y) = 0$$

$$\Rightarrow \quad x\sin(a+y) = -\sin a\cdot\cos(a+y)$$

$$\Rightarrow \quad x = \frac{-\sin a\cdot\cos(a+y)}{\sin(a+y)}$$

Now, differentiating Eq. (i) w.r.t. $y$, we get

$$\frac{d}{dy}(ax^2) + \frac{d}{dy}(2hxy) + \frac{d}{dy}(by^2) + \frac{d}{dy}(2gx) + \frac{d}{dy}(2fy) + \frac{d}{dy}(c) = 0$$

$$\Rightarrow \quad a \cdot 2x \cdot \frac{dx}{dy} + 2h \cdot \left( x \cdot \frac{d}{dy} y + y \cdot \frac{d}{dy} x \right) + b \cdot 2y + 2g \cdot \frac{dx}{dy} + 2f + 0 = 0$$

$$\Rightarrow \quad \frac{dx}{dy}[2ax + 2hy + 2g] = -2hx - 2by - 2f$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-2(hx + by + f)}{2(ax + hy + g)} = \frac{-(hx + by + f)}{(ax + hy + g)} \quad \ldots \text{(iii)}$$

$$\therefore \quad \frac{dy}{dx} \cdot \frac{dx}{dy} = \frac{-(ax + hy + g)}{(hx + by + f)} \cdot \frac{-(hx + by + f)}{(ax + hy + g)} \quad \text{[using Eqs. (ii) and (iii)]}$$

$$= 1 = \text{RHS}$$

**Hence proved.**

## Q. 59 If $x = e^{x/y}$, then prove that $\frac{dy}{dx} = \frac{x - y}{x \log x}$.

**Sol.** We have,

$$x = e^{x/y}$$

$$\therefore \quad \frac{d}{dx} x = \frac{d}{dx} e^{x/y}$$

$$\Rightarrow \quad 1 = e^{x/y} \cdot \frac{d}{dx}(x / y)$$

$$\Rightarrow \quad 1 = e^{x/y} \cdot \left[ \frac{y \cdot 1 - x \cdot dy/dx}{y^2} \right]$$

$$\Rightarrow \quad y^2 = y \cdot e^{x/y} - x \cdot \frac{dy}{dx} \cdot e^{x/y}$$

$$\Rightarrow \quad x \cdot \frac{dy}{dx} \cdot e^{x/y} = ye^{x/y} - y^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{y(e^{x/y} - y)}{x \cdot e^{x/y}}$$

$$= \frac{(e^{x/y} - y)}{e^{x/y} \cdot \frac{x}{y}} \quad \left[ \because x = e^{x/y} \Rightarrow \log x = \frac{x}{y} \right]$$

$$= \frac{x - y}{x \cdot \log x}$$

**Hence proved.**

## Q. 60 If $y^x = e^{y - x}$, then prove that $\frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$.

**Sol.** We have,

$$y^x = e^{y - x}$$

$$\Rightarrow \quad \log y^x = \log e^{y - x}$$

$$\Rightarrow \quad x \log y = y - x \cdot \log_e = (y - x) \quad [\because \log_e = 1]$$

$$\Rightarrow \quad \log y = \frac{(y - x)}{x} \quad \ldots \text{(i)}$$

Now, differentiating w.r.t. $x$, we get

$$\frac{d}{dy} \log y \cdot \frac{dy}{dx} = \frac{d}{dx} \frac{(y - x)}{x}$$

**Q. 56** $\tan^{-1}(x^2+y^2)=a$

**Sol.** We have, $\tan^{-1}(x^2+y^2)=a$

On differentiating both sides w.r.t. $x$, we get

$$\frac{d}{dx}\tan^{-1}(x^2+y^2)=\frac{d}{dx}(a)$$

$$\Rightarrow \quad \frac{1}{1+(x^2+y^2)^2}\cdot\frac{d}{dx}(x^2+y^2)=0$$

$$\Rightarrow \quad 2x+\frac{d}{dy}y^2\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad 2y\cdot\frac{dy}{dx}=-2x$$

$$\therefore \quad \frac{dy}{dx}=\frac{-2x}{2y}=\frac{-x}{y}$$

**Q. 57** $(x^2+y^2)^2=xy$

**Sol.** We have, $(x^2+y^2)^2=xy$

On differentiating both sides w.r.t. $x$, we get

$$\frac{d}{dx}(x^2+y^2)^2=\frac{d}{dx}(xy)$$

$$\Rightarrow \quad 2(x^2+y^2)\cdot\frac{d}{dx}(x^2+y^2)=x\cdot\frac{d}{dx}y+y\cdot\frac{d}{dx}x$$

$$\Rightarrow \quad 2(x^2+y^2)\cdot\left(2x+2y\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$$

$$\Rightarrow \quad 2x^2\cdot 2x+2x^2\cdot 2y\frac{dy}{dx}+2y^2\cdot 2x+2y^2\cdot 2y\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\Rightarrow \quad \frac{dy}{dx}[4x^2y+4y^3-x]=y-4x^3-4xy^2$$

$$\therefore \quad \frac{dy}{dx}=\frac{(y-4x^3-4xy^2)}{(4x^2y+4y^3-x)}$$

**Q. 58** If $ax^2+2hxy+by^2+2gx+2fy+c=0$, then show that $\frac{dy}{dx}\cdot\frac{dx}{dy}=1$.

**Sol.** We have, $ax^2+2hxy+by^2+2gx+2fy+c=0$ ...(i)

On differentiating both sides w.r.t. $x$, we get

$$\frac{d}{dx}(ax^2)+\frac{d}{dx}(2hxy)+\frac{d}{dx}(by^2)+\frac{d}{dx}(2gx)+\frac{d}{dx}(2fy)+\frac{d}{dx}(c)=0$$

$$\Rightarrow \quad 2ax+2h\left(x\cdot\frac{dy}{dx}+y\cdot 1\right)+b\cdot 2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}+0=0$$

$$\Rightarrow \quad \frac{dy}{dx}[2hx+2by+2f]=-2ax-2hy-2g$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-2(ax+hy+g)}{2(hx+by+f)}$$

$$=\frac{-(ax+hy+g)}{(hx+by+f)} \qquad \text{...(ii)}$$

*Find $\dfrac{dy}{dx}$ when x and y are connected by the relation given.*

## Q. 54 $\sin(xy) + \dfrac{x}{y} = x^2 - y$

**Sol.** We have, $$\sin(xy) + \frac{x}{y} = x^2 - y$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}x^2 - \frac{d}{dx}y$$

$$\Rightarrow \quad \cos xy \cdot \frac{d}{dx}(xy) + \frac{y\dfrac{d}{dx}x - x\cdot\dfrac{d}{dx}y}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad \cos xy \cdot \left[x \cdot \frac{d}{dx}y + y \cdot \frac{d}{dx}\cdot x\right] + \frac{y - x\dfrac{dy}{dx}}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad x\cos xy \cdot \frac{dy}{dx} + y\cos xy + \frac{y}{y^2} - \frac{x}{y^2}\frac{dy}{dx} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[x\cos xy - \frac{x}{y^2} + 1\right] = 2x - y\cos xy - \frac{y}{y^2}$$

$$\therefore \quad \frac{dy}{dx} = \left[\frac{2xy - y^2\cos xy - 1}{y}\right]\left[\frac{y^2}{xy^2\cos xy - x + y^2}\right]$$

$$= \frac{(2xy - y^2\cos xy - 1)\,y}{(xy^2\cos xy - x + y^2)}$$

## Q. 55 $\sec(x + y) = xy$

**Sol.** We have, $\sec(x + y) = xy$

On differentiating both sides w.r.t. $x$, we get

$$\frac{d}{dx}\sec(x + y) = \frac{d}{dx}(xy)$$

$$\Rightarrow \quad \sec(x + y)\cdot\tan(x + y)\cdot\frac{d}{dx}(x + y) = x\cdot\frac{d}{dx}y + y\cdot\frac{d}{dx}x$$

$$\Rightarrow \quad \sec(x + y)\cdot\tan(x + y)\cdot\left(1 + \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \sec(x + y)\tan(x + y) + \sec(x + y)\cdot\tan(x + y).\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \frac{dy}{dx}[\sec(x + y)\cdot\tan(x + y) - x] = y - \sec(x + y)\cdot\tan(x + y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{y - \sec(x + y)\cdot\tan(x + y)}{\sec(x + y)\cdot\tan(x + y) - x}$$

**Q. 52** Differentiate $\dfrac{x}{\sin x}$ w.r.t. $\sin x$.

**Sol.** Let
$$u = \frac{x}{\sin x} \text{ and } v = \sin x$$

$$\therefore \quad \frac{du}{dx} = \frac{\sin x \cdot \frac{d}{dx} x - x \cdot \frac{d}{dx} \sin x}{(\sin x)^2}$$

$$= \frac{\sin x - x \cos x}{\sin^2 x} \quad \text{...(i)}$$

and
$$\frac{dv}{dx} = \frac{d}{dx} \sin x = \cos x \quad \text{...(ii)}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\sin x - x \cos x / \sin^2 x}{\cos x}$$

$$= \frac{\sin x - x \cos x}{\sin^2 x \cos x} = \frac{\dfrac{\sin x - x \cos x}{\cos x}}{\dfrac{\sin^2 x \cos x}{\cos x}}$$

[dividing by $\cos x$ in both numerator and denominator]

$$= \frac{\tan x - x}{\sin^2 x}$$

**Q. 53** Differentiate $\tan^{-1} \dfrac{\sqrt{1 + x^2} - 1}{x}$ w.r.t. $\tan^{-1} x$, when $x \neq 0$.

**Sol.** Let
$$u = \tan^{-1} \frac{\sqrt{1 + x^2} - 1}{x} \text{ and } v = \tan^{-1} x$$

$$\therefore \quad x = \tan \theta$$

$$\Rightarrow \quad u = \tan^{-1} \frac{\sqrt{1 + \tan^2 \theta} - 1}{\tan \theta}$$

$$= \tan^{-1} \frac{(\sec \theta - 1) \cos \theta}{\sin \theta}$$

$$= \tan^{-1} \left( \frac{1 - \cos \theta}{\sin \theta} \right)$$

$$= \tan^{-1} \left[ \frac{1 - 1 + 2\sin^2 \theta/2}{2\sin \theta/2 \cdot \cos \theta/2} \right] \quad [\because \cos \theta = 1 - 2\sin^2 \theta]$$

$$= \tan^{-1} \left[ \tan \frac{\theta}{2} \right]$$

$$= \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{2} \frac{d}{dx} \tan^{-1} x = \frac{1}{2} \cdot \frac{1}{1 + x^2} \quad \text{...(i)}$$

and
$$\frac{dv}{dx} = \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \quad \text{...(ii)}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{1/2\,(1 + x^2)}{1/(1 + x^2)} = \frac{(1 + x^2)}{2(1 + x^2)} = \frac{1}{2}$$

$$= a\left[\sin 2t\cdot(-\sin 2t)\cdot\frac{d}{dt}2t + (1+\cos 2t)\cdot\cos 2t\cdot\frac{d}{dt}2t\right]$$

$$= -2a\sin^2 2t + 2a\cos 2t\,(1+\cos 2t)$$

$$\Rightarrow \quad \frac{dx}{dt} = -2a\,[\sin^2 2t - \cos 2t\,(1+\cos 2t)] \quad \text{...(i)}$$

and
$$\frac{dy}{dt} = b\left[\cos 2t\cdot\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t)\cdot\frac{d}{dt}\cos 2t\right]$$

$$= b\left[\cos 2t\cdot(\sin 2t)\frac{d}{dt}2t + (1-\cos 2t)(-\sin 2t)\cdot\frac{d}{dt}2t\right]$$

$$= b\,[2\sin 2t\cdot\cos 2t + 2\,(1-\cos 2t)(-\sin 2t)]$$

$$= 2b\,[\sin 2t\cdot\cos 2t - (1-\cos 2t)\sin 2t] \quad \text{...(ii)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2b\,[-\sin 2t\cdot\cos 2t + (1-\cos 2t)\sin 2t]}{-2a\,[\sin^2 2t - \cos 2t\,(1+\cos 2t)]}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{t=\pi/4} = \frac{b}{a}\,\frac{\left[-\sin\frac{\pi}{2}\cos\frac{\pi}{2} + \left(1-\cos\frac{\pi}{2}\right)\sin\frac{\pi}{2}\right]}{\left[\sin^2\frac{\pi}{2} - \cos\frac{\pi}{2}\left(1+\cos\frac{\pi}{2}\right)\right]}$$

$$= \frac{b}{a}\cdot\frac{(0+1)}{(1-0)} \qquad \left[\because \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{\pi}{2} = 0\right]$$

$$= \frac{b}{a}$$

**Hence proved.**

**Q. 51** If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, then find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.

**Sol.** $\because$
$$x = 3\sin t - \sin 3t \text{ and } y = 3\cos t - \cos 3t$$

$$\therefore \quad \frac{dx}{dt} = 3\cdot\frac{d}{dt}\sin t - \frac{d}{dt}\sin 3t$$

$$= 3\cos t - \cos 3t\cdot\frac{d}{dt}3t = 3\cos t - 3\cos 3t \quad \text{...(i)}$$

and
$$\frac{dy}{dt} = 3\cdot\frac{d}{dt}\cos t - \frac{d}{dt}\cos 3t$$

$$= -3\sin t + \sin 3t\cdot\frac{d}{dt}3t$$

$$\frac{dy}{dt} = 3\sin 3t - 3t\sin t \quad \text{...(ii)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(\sin 3t - \sin t)}{3(\cos t - \cos 3t)}$$

Now,
$$\left(\frac{dy}{dx}\right)_{t=\pi/3} = \frac{\sin\frac{3\pi}{3} - \sin\frac{\pi}{3}}{\left(\cos\frac{\pi}{3} - \cos 3\frac{\pi}{3}\right)} = \frac{0-\sqrt{3}/2}{\frac{1}{2}-(-1)}$$

$$= \frac{-\sqrt{3}/2}{3/2} = \frac{-\sqrt{3}}{3} = \frac{-1}{\sqrt{3}}$$

$$= \frac{t^2 \cdot \frac{1}{t} - (1 + \log t) \cdot 2t}{t^4} = \frac{t - (1 + \log t) \cdot 2t}{t^4}$$

$$= \frac{t}{t^4} [1 - 2(1 + \log t) = \frac{-1 - 2 \log t}{t^3} \quad \text{... (i)}$$

and $\quad \frac{dy}{dt} = \frac{t \cdot \frac{d}{dt}(3 + 2 \log t) - (3 + 2 \log t) \cdot \frac{d}{dt} t}{t^2}$

$$= \frac{t \cdot 2 \cdot \frac{1}{t} - (3 + 2 \log t) \cdot 1}{t^2}$$

$$= \frac{2 - 3 - 2 \log t}{t^2} = \frac{-1 - 2 \log t}{t^2} \quad \text{...(ii)}$$

$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1 - 2 \log t / t^2}{-1 - 2 \log t / t^3} = t$

**Q. 49** If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, then prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.

**Sol.** $\because \quad x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

$\therefore \quad \frac{dx}{dt} = \frac{d}{dt} e^{\cos 2t} = e^{\cos 2t} \cdot \frac{d}{dt} \cos 2t$

$$= e^{\cos 2t} \cdot (-\sin 2t) \cdot \frac{d}{dt}(2t)$$

$$\frac{dx}{dt} = -2 e^{\cos 2t} \cdot \sin 2t \quad \text{...(i)}$$

and $\quad \frac{dy}{dt} = \frac{d}{dt} e^{\sin 2t} = e^{\sin 2t} \cdot \frac{d}{dt} \sin 2t$

$$= e^{\sin 2t} \cos 2t \cdot \frac{d}{dt} 2t$$

$$= 2e^{\sin 2t} \cdot \cos 2t \quad \text{...(ii)}$$

$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{\sin 2t} \cdot \cos 2t}{-2e^{\cos 2t} \cdot \sin 2t}$

$$= \frac{e^{\sin 2t} \cdot \cos 2t}{e^{\cos 2t} \cdot \sin 2t} \quad \text{...(iii)}$$

We know that, $\quad \log x = \cos 2t \cdot \log e = \cos 2t \quad \text{...(iv)}$

and $\quad \log y = \sin 2t \cdot \log e = \sin 2t \quad \text{...(v)}$

$\therefore \quad \frac{dy}{dx} = \frac{-y \log x}{x \log y}$

[using Eqs. (iv) and (v) in Eq. (iii) and $x = e^{\cos 2t}$, $y = e^{\sin 2t}$]

**Hence proved.**

**Q. 50** If $x = a \sin 2t (1 + \cos 2t)$ and $y = b \cos 2t (1 - \cos 2t)$, then show that $\left(\frac{dy}{dx}\right)_{t = \pi/4} = \frac{b}{a}$.

**Sol.** $\because \quad x = a \sin 2t (1 + \cos 2t)$ and $y = b \cos 2t (1 - \cos 2t)$

$\therefore \quad \frac{dx}{dt} = a \left[\sin 2t \cdot \frac{d}{dt}(1 + \cos 2t) + (1 + \cos 2t) \cdot \frac{d}{dt} \sin 2t\right]$

**Q. 47** $\sin x = \dfrac{2t}{1+t^2}$, $\tan y = \dfrac{2t}{1-t^2}$

**Sol.** $\because$
$$\sin x = \frac{2t}{1+t^2} \qquad \text{...(i)}$$
and
$$\tan y = \frac{2t}{1-t^2} \qquad \text{...(ii)}$$
$$\therefore \quad \frac{d}{dx}\sin x \cdot \frac{dx}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$
$$\Rightarrow \quad \cos x \frac{dx}{dt} = \frac{(1+t^2)\cdot \frac{d}{dt}(2t) - (2t)\cdot \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$
$$= \frac{2\,(1+t^2) - 2t \cdot 2t}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$
$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\cos x}$$
$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\sin^2 x}} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$
$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{(1+t^2)}{(1-t^2)} = \frac{2}{1+t^2} \qquad \text{...(iii)}$$
Also,
$$\frac{d}{dy}\tan y \cdot \frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$
$$\sec^2 y \frac{dy}{dt} = \frac{(1-t^2)\frac{d}{dt}\cdot(2t) - 2t\cdot\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$
$$\frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2}\cdot\frac{1}{\sec^2 y}$$
$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{(1+\tan^2 y)} = \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{1+\frac{4t^2}{(1-t^2)^2}}$$
$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{(1-t^2)^2}{(1+t^2)^2} = \frac{2}{1+t^2} \qquad \text{...(iv)}$$
$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2/1+t^2}{2/1+t^2} = 1 \qquad \text{[from Eqs. (iii) and (iv)]}$$

**Q. 48** $x = \dfrac{1+\log t}{t^2}$, $y = \dfrac{3+2\log t}{t}$

**Sol.** $\because$
$$x = \frac{1+\log t}{t^2} \text{ and } y = \frac{3+2\log t}{t}$$
$$\therefore \quad \frac{dx}{dt} = \frac{t^2.\frac{d}{dt}(1+\log t) - (1+\log t).\frac{d}{dt}t^2}{(t^2)^2}$$

**Q. 45** $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

**Sol.** $\because$ $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ and $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

$\therefore$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[e^{\theta}\cdot\left(\theta + \frac{1}{\theta}\right)\right]$$
$$= e^{\theta}\cdot\frac{d}{d\theta}\left(\theta + \frac{1}{\theta}\right) + \left(\theta + \frac{1}{\theta}\right)\cdot\frac{d}{d\theta}e^{\theta}$$
$$= e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \left(\theta + \frac{1}{\theta}\right)e^{\theta}$$
$$= e^{\theta}\left(1 - \frac{1}{\theta^2} + \theta + \frac{1}{\theta}\right)$$
$$= e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right) \quad \text{...(i)}$$

and
$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[e^{-\theta}\cdot\left(\theta - \frac{1}{\theta}\right)\right]$$
$$= e^{-\theta}\cdot\frac{d}{d\theta}\left(\theta - \frac{1}{\theta}\right) + \frac{d}{d\theta}e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$
$$= e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right)e^{-\theta}\cdot\frac{d}{d\theta}(-\theta)$$
$$= e^{-\theta}\left[\frac{\theta^2 + 1}{\theta^2} - \frac{\theta^2 - 1}{\theta}\right] = e^{-\theta}\left[\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right] \quad \text{...(ii)}$$

$\therefore$
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)}{e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)}$$
$$= e^{-2\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1}\right)$$

**Q. 46** $x = 3\cos\theta - 2\cos^3\theta,\ y = 3\sin\theta - 2\sin^3\theta$

**Sol.** $\because$ $x = 3\cos\theta - 2\cos^3\theta$ and $y = 3\sin\theta - 2\sin^3\theta$

$\therefore$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}(3\cos\theta) - \frac{d}{d\theta}(2\cos^3\theta)$$
$$= 3\cdot(-\sin\theta) - 2\cdot 3\cos^2\theta\cdot\frac{d}{d\theta}\cdot\cos\theta$$
$$= -3\sin\theta + 6\cos^2\theta\sin\theta$$

and
$$\frac{dy}{d\theta} = 3\cos A - 2\cdot 3\sin^2\theta\cdot\frac{d}{d\theta}\cdot\sin\theta$$
$$= 3\cos\theta - 6\sin^2\theta\cdot\cos\theta$$

Now,
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta\sin\theta}$$
$$= \frac{3\cos\theta(1 - 2\sin^2\theta)}{3\sin\theta(-1 + 2\cos^2\theta)} = \cot\theta\cdot\frac{\cos 2\theta}{\cos 2\theta} = \cot\theta$$

**Q. 43** $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$, $-1< x< 1$, $x \neq 0$

**Sol.** Let $$y=\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$$

Put $$x^2=\cos 2\theta$$

$\therefore$
$$y=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$
$$=\tan^{-1}\left(\frac{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}\right)$$
$$=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)=\tan^{-1}\left[\frac{\sqrt{2}(\cos\theta+\sin\theta)}{\sqrt{2}(\cos\theta-\sin\theta)}\right]$$
$$=\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)=\tan^{-1}\left(\frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}\right)$$
$$=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$
$$=\tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right) \qquad \left[\because \tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\cdot\tan b}\right]$$
$$=\frac{\pi}{4}+\theta=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \qquad \left[\because 2\theta=\cos^{-1}x^2 \Rightarrow \theta=\frac{1}{2}\cos^{-1}x^2\right]$$

$\therefore$
$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{d}{dx}\left(\frac{1}{2}\cos^{-1}x^2\right)$$
$$=0+\frac{1}{2}\cdot\frac{-1}{\sqrt{1-x^4}}\cdot\frac{d}{dx}x^2=\frac{1}{2}\cdot\frac{-2x}{\sqrt{1-x^4}}=\frac{-x}{\sqrt{1-x^4}}$$

*Find $\dfrac{dy}{dx}$ of each of the functions expressed in parametric form.*

**Q. 44** $x=t+\dfrac{1}{t}$, $y=t-\dfrac{1}{t}$

**Sol.** $\because$ $$x=t+\frac{1}{t} \text{ and } y=t-\frac{1}{t}$$

$\therefore$ $$\frac{dx}{dt}=\frac{d}{dt}\left(t+\frac{1}{t}\right) \text{ and } \frac{dy}{dt}=\frac{d}{dt}\left(t-\frac{1}{t}\right)$$

$\Rightarrow$ $$\frac{dx}{dt}=1+(-1)t^{-2} \text{ and } \frac{dy}{dt}=1-(-1)t^{-2}$$

$\Rightarrow$ $$\frac{dx}{dt}=1-\frac{1}{t^2} \text{ and } \frac{dy}{dt}=1+\frac{1}{t^2}$$

$\Rightarrow$ $$\frac{dx}{dt}=\frac{t^2-1}{t^2} \text{ and } \frac{dy}{dt}=\frac{t^2+1}{t^2}$$

$\therefore$ $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{t^2+1/t^2}{t^2-1/t^2}=\frac{t^2+1}{t^2-1}$$

$$= \tan^{-1}\frac{a}{b} - x$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right) - \frac{d}{dx}(x)$$

$$= 0 - 1 \qquad \left[\because \frac{d}{dx}\left(\frac{a}{b}\right) = 0\right]$$

$$= -1$$

**Q. 41** $\sec^{-1}\left(\frac{1}{4x^3 - 3x}\right), 0 < x < \frac{1}{\sqrt{2}}$

**Sol.** Let $y = \sec^{-1}\left(\frac{1}{4x^3 - 3x}\right)$ ...(i)

On putting $x = \cos\theta$ in Eq. (i), we get

$$y = \sec^{-1}\frac{1}{4\cos^3\theta - 3\cos\theta}$$

$$= \sec^{-1}\frac{1}{\cos 3\theta}$$

$$= \sec^{-1}(\sec 3\theta) = 3\theta$$

$$= 3\cos^{-1}x \qquad [\because \theta = \cos^{-1}x]$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(3\cos^{-1}x)$$

$$= 3\cdot\frac{-1}{\sqrt{1-x^2}}$$

**Q. 42** $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), \frac{-1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$

**Sol.** Let $y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$

Put $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$$\therefore \quad y = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right] \qquad \left[\because \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right]$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta$$

$$= 3\tan^{-1}\frac{x}{a} \qquad \left[\because \theta = \tan^{-1}\frac{x}{a}\right]$$

$$\therefore \quad \frac{dy}{dx} = 3\cdot\frac{d}{dx}\tan^{-1}\frac{x}{a} = 3\cdot\left[\frac{1}{1+\frac{x^2}{a^2}}\right]\cdot\frac{d}{dx}\cdot\left(\frac{x}{a}\right)$$

$$= 3\cdot\frac{a^2}{a^2+x^2}\cdot\frac{1}{a} = \frac{3a}{a^2+x^2}$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2}\left[\frac{(1-\cos x)^2}{\sin x}\right]^{-1/2} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2} \cdot \frac{\sin x}{(1-\cos x)} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin^2 x}{4(1+\cos x)(1-\cos x)} = \frac{1}{2} \cdot \frac{\sin^2 x}{(1-\cos^2 x)}$$

$$= \frac{1}{2} \cdot \frac{\sin^2 x}{\sin^2 x} = \frac{1}{2}$$

**Alternate Method**

Let $$y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{1-1+2\sin^2 \frac{x}{2}}{1+2\cos^2 \frac{x}{2}-1}}\right) \qquad \left[\because \cos x = 1-2\sin^2\frac{x}{2} = 2\cos^2\frac{x}{2}-1\right]$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

On differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{2}$$

## Q. 39 $\tan^{-1}(\sec x + \tan x),\ \frac{-\pi}{2} < x < \frac{\pi}{2}$

**Sol.** Let $y = \tan^{-1}(\sec x + \tan x)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}(\sec x + \tan x)$$

$$= \frac{1}{1+(\sec x+\tan x)^2} \cdot \frac{d}{dx}(\sec x + \tan x) \qquad \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

$$= \frac{1}{1+\sec^2 x + \tan^2 x + 2\sec x \cdot \tan x} \cdot [\sec x \cdot \tan x + \sec^2 x]$$

$$= \frac{1}{(\sec^2 x + \sec^2 x + 2\sec x \cdot \tan x)} \cdot \sec x \cdot (\sec x + \tan x)$$

$$= \frac{1}{2\sec x(\tan x + \sec x)} \cdot \sec x(\sec x + \tan x) \quad = \frac{1}{2}$$

## Q. 40 $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right),\ \frac{-\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1.$

**Sol.** Let $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

$$= \tan^{-1}\left[\frac{\frac{a\cos x}{b\cos x} - \frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x} + \frac{a\sin x}{b\cos x}}\right] = \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1+\frac{a}{b}\tan x}\right]$$

$$= \tan^{-1}\frac{a}{b} - \tan^{-1}\tan x \qquad \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

**Q. 37** $\cos^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right), -\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

**Sol.** Let $$y=\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$\therefore$ $$\frac{dy}{dx}=\frac{d}{dx}\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$=\frac{-1}{\sqrt{1-\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right)^2}}\cdot\frac{d}{dx}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$\left[\because \frac{d}{dx}(\cos x)=-\frac{1}{\sqrt{1-x^2}}\right]$$

$$=\frac{-1}{\sqrt{4-\dfrac{(\sin^2 x+\cos^2 x+2\sin x\cdot\cos x)}{2}}}\cdot\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$=\frac{-1\cdot\sqrt{2}}{\sqrt{1-\sin 2x}}\cdot\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$[\because 1-\sin 2x=(\cos x-\sin x)^2=\cos^2 x+\sin^2 x-2\sin x\cos x]$$

$$=\frac{-1(\cos x-\sin x)}{(\cos x-\sin x)}=-1$$

**Q. 38** $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}, -\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

**Sol.** Let $$y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$\therefore$ $$\frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$=\frac{1}{1+\sqrt{\left(\dfrac{1-\cos x}{1+\cos x}\right)^2}}\cdot\frac{d}{dx}\left[\frac{1-\cos x}{1+\cos x}\right]^{1/2}\qquad\left[\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right]$$

$$=\frac{1}{1+\dfrac{1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{1-\cos x}{1+\cos x}\right]^{-1/2}\cdot\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$=\frac{1}{\dfrac{1+\cos x+1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{(1-\cos x)}{(1+\cos x)}\cdot\frac{(1-\cos x)}{(1-\cos x)}\right]^{-1/2}$$

$$\cdot\frac{(1+\cos x)\cdot\sin x+(1-\cos x)\cdot\sin x}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x\,(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x\,(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

**Q. 35** $\sin^m x \cdot \cos^n x$

**Sol.** Let $y = \sin^m x \cdot \cos^n x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[(\sin x)^m \cdot (\cos x)^n\right]$$

$$= (\sin x)^m \cdot \frac{d}{dx}(\cos x)^n + (\cos x)^n \cdot \frac{d}{dx}(\sin x)^m$$

$$= (\sin x)^m \cdot n(\cos x)^{n-1} \cdot \frac{d}{dx}\cos x + (\cos x)^n\ m(\sin x)^{m-1} \cdot \frac{d}{dx}\sin x$$

$$= (\sin x)^m \cdot n(\cos x)^{n-1}\ (-\sin x) + (\cos x)^n \cdot m(\sin x)^{m-1}\cos x$$

$$= -n\sin^m x \cdot \cos^{n-1} x \cdot (\sin x) + m\cos^n x \cdot \sin^{m-1} x \cdot \cos x$$

$$= -n \cdot \sin^m x \cdot \sin x \cdot \cos^n x \cdot \frac{1}{\cos x} + m \cdot \sin^m x . \frac{1}{\sin x} \cdot \cos^n x \cdot \cos x$$

$$= -n \cdot \sin^m x \cdot \cos^n x \cdot \tan x + m\sin^m x \cdot \cos^n x \cdot \cot x$$

$$= \sin^m x \cdot \cos^n x\,[-n\tan x + m\cot x]$$

**Q. 36** $(x+1)^2(x+2)^3(x+3)^4$

**Sol.** Let $y = (x+1)^2\,(x+2)^3\,(x+3)^4$

$$\therefore \quad \log y = \log\{(x+1)^2 \cdot (x+2)^3(x+3)^4\}$$

$$= \log(x+1)^2 + \log(x+2)^3 + \log(x+3)^4$$

and $$\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}[2\log(x+1)] + \frac{d}{dx}[3\log(x+2)] + \frac{d}{dx}[4\log(x+3)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{(x+1)} \cdot \frac{d}{dx}(x+1) + 3 \cdot \frac{1}{(x+2)} \cdot \frac{d}{dx}(x+2) + 4 \cdot \frac{1}{(x+3)} \cdot \frac{d}{dx}(x+3) \quad \left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= \left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)}\right]$$

$$= (x+1)^2 \cdot (x+2)^3 \cdot (x+3)^4\left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)}\right]$$

$$= (x+1)^2 \cdot (x+2)^3 \cdot (x+3)^4 \left[\frac{2(x+2)(x+3) + 3(x+1)(x+3) + 4(x+1)(x+2)}{(x+1)(x+2)(x+3)}\right]$$

$$= \frac{(x+1)^2\,(x+2)^3\,(x+3)^4}{(x+1)(x+2)(x+3)}\,[2(x^2+5x+6) + 3(x^2+4x+3) + 4(x^2+3x+2)]$$

$$= (x+1)(x+2)^2\,(x+3)^3\,[2x^2 + 10x + 12 + 3x^2 + 12x + 9 + 4x^2 + 12x + 8]$$

$$= (x+1)(x+2)^2\,(x+3)^3\,[9x^2 + 34x + 29]$$

**Q. 32** $\sin x^2 + \sin^2 x + \sin^2 (x^2)$

**Sol.** Let
$$y = \sin x^2 + \sin^2 x + \sin^2 (x^2)$$
$\therefore$
$$\frac{dy}{dx} = \frac{d}{dx}\sin (x^2) + \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\sin x^2)^2$$
$$= \cos (x^2)\frac{d}{dx}(x^2) + 2\sin x \cdot \frac{d}{dx}\sin x + 2\sin x^2 \cdot \frac{d}{dx}\sin x^2$$
$$= \cos x^2 2x + 2\cdot \sin x \cdot \cos x + 2\sin x^2 \cos x^2 \cdot \frac{d}{dx}x^2$$
$$= 2x\cos (x)^2 + 2\cdot \sin x \cdot \cos x + 2\sin x^2 \cdot \cos x^2 \cdot 2x$$
$$= 2x\cos (x)^2 + \sin 2x + \sin 2(x)^2 \cdot 2x$$
$$= 2x\cos (x^2) + 2x \cdot \sin 2(x^2) + \sin 2x$$

**Q. 33** $\sin^{-1}\frac{1}{\sqrt{x+1}}$

**Sol.** Let
$$y = \sin^{-1}\frac{1}{\sqrt{x+1}}$$
$\therefore$
$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\frac{1}{\sqrt{x+1}}$$
$$= \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}} \cdot \frac{d}{dx}\frac{1}{(x+1)^{1/2}} \qquad \left[\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$$
$$= \frac{1}{\sqrt{\frac{x+1-1}{x+1}}} \cdot \frac{d}{dx} \cdot (x+1)^{-1/2}$$
$$= \sqrt{\frac{x+1}{x}} \cdot \frac{-1}{2}(x+1)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}(x+1)$$
$$= \frac{(x+1)^{1/2}}{x^{1/2}} \cdot \left(-\frac{1}{2}\right)(x+1)^{-3/2} = \frac{-1}{2\sqrt{x}} \cdot \left(\frac{1}{x+1}\right)$$

**Q. 34** $(\sin x)^{\cos x}$

**Sol.** Let
$$y = (\sin x)^{\cos x}$$
$\Rightarrow$
$$\log y = \log (\sin x)^{\cos x} = \cos x \log \sin x$$
$\therefore$
$$\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}(\cos x \cdot \log \sin x)$$
$\Rightarrow$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x \cdot \frac{d}{dx}\log \sin x + \log \sin x \cdot \frac{d}{dx}\cos x$$
$$= \cos x \cdot \frac{1}{\sin x}\cdot\frac{d}{dx}\sin x + \log \sin x \cdot (-\sin x)$$
$$= \cot x \cdot \cos x - \log (\sin x)\cdot \sin x \qquad \left[\because \cot x = \frac{\cos x}{\sin x}\right]$$
$\therefore$
$$\frac{dy}{dx} = y\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log (\sin x)\right]$$
$$= \sin x^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log (\sin x)\right]$$

**Q. 28** $\log[\log\ (\log x^5)]$

**Sol.** Let $y = \log[\log(\log x^5)]$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\log\ (\log\ \log x^5)]$$

$$= \frac{1}{\log\log x^5} \cdot \frac{d}{dx}(\log \cdot \log x^5)$$

$$= \frac{1}{\log\log x^5} \cdot \left(\frac{1}{\log x^5}\right) \cdot \frac{d}{dx}\log x^5$$

$$= \frac{1}{\log\log x^5} \cdot \frac{1}{\log x^5} \cdot \frac{d}{dx}(5\log x) = \frac{5}{x \cdot \log(\log x^5) \cdot \log(x^5)}$$

**Q. 29** $\sin\sqrt{x} + \cos^2\sqrt{x}$

**Sol.** Let $y = \sin\sqrt{x} + (\cos\sqrt{x})^2$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\sin(x^{1/2}) + \frac{d}{dx}[\cos(x^{1/2})]^2$$

$$= \cos x^{1/2} \cdot \frac{d}{dx}x^{1/2} + 2\cos(x^{1/2})\ \frac{d}{dx}[\cos(x^{1/2})]$$

$$= \cos(x^{1/2})\ \frac{1}{2}x^{-1/2} + 2 \cdot \cos(x^{1/2}) \cdot \left[-\sin(x^{1/2}) \cdot \frac{d}{dx}x^{1/2}\right]$$

$$= \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}[-2\cos(x^{1/2})] \cdot \sin x^{1/2} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}[\cos(\sqrt{x}) - \ \sin(2\sqrt{x})]$$

**Q. 30** $\sin^n(ax^2 + bx + c)$

**Sol.** Let $y = \sin^n(ax^2 + bx + c)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\sin(ax^2 + bx + c)]^n$$

$$= n \cdot [\sin(ax^2 + bx + c)]^{n-1} \cdot \frac{d}{dx}\sin(ax^2 + bx + c)$$

$$= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot \frac{d}{dx}(ax^2 + bx + c)$$

$$= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot (2ax + b)$$

$$= n \cdot (2ax + b) \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c)$$

**Q. 31** $\cos(\tan\sqrt{x+1})$

**Sol.** Let $y = \cos(\tan\sqrt{x+1})$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\cos(\tan\sqrt{x+1}) = -\sin(\tan\sqrt{x+1}) \cdot \frac{d}{dx}(\tan\sqrt{x+1})$$

$$= -\sin(\tan\sqrt{x+1}) \cdot \sec^2\sqrt{x+1} \cdot \frac{d}{dx}(x+1)^{1/2} \qquad \left[\because \frac{d}{dx}(\tan x) = \sec^2 x\right]$$

$$= -\sin(\tan\sqrt{x+1}) \cdot (\sec\sqrt{x+1})^2 \cdot \frac{1}{2}(x+1)^{-1/2} \cdot \frac{d}{dx}(x+1)$$

$$= \frac{-1}{2\sqrt{x+1}} \cdot \sin(\tan\sqrt{x+1}) \cdot \sec^2(\sqrt{x+1})$$

**Q. 25** $2^{\cos^2 x}$

**Sol.** Let $$y = 2^{\cos^2 x}$$

$\therefore$ $$\log y = \log 2^{\cos^2 x} = \cos^2 x \cdot \log 2$$

On differentiating w.r.t. $x$, we get

$$\frac{d}{dy}\log y . \frac{dy}{dx} = \frac{d}{dx}\log 2 \cdot \cos^2 x$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \log 2\frac{d}{dx}(\cos x)^2$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \log 2\cdot[2\cos x]\cdot\frac{d}{dx}\cos x$$

$$= \log 2 \cdot 2\cos x \cdot (-\sin x)$$

$$= \log 2 \cdot [-(\sin 2x)]$$

$\therefore$ $$\frac{dy}{dx} = -y \cdot \log 2\,(\sin 2x)$$

$$= -2^{\cos^2 x}\cdot \log 2\,(\sin 2x)$$

**Q. 26** $\dfrac{8^x}{x^8}$

**Sol.** Let $$y = \frac{8^x}{x^8} \Rightarrow \log y = \log\frac{8^x}{x^8}$$

$\Rightarrow$ $$\frac{d}{dy}\log y\cdot\frac{dy}{dx} = \frac{d}{dx}[\log 8^x - \log x^8]$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}[x\cdot\log 8 - 8\cdot\log x]$$

On differentiating w.r.t. $x$, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log 8\cdot 1 - 8\cdot\frac{1}{x}$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \log 8 - \frac{8}{x}$$

$\therefore$ $$\frac{dy}{dx} = y\left(\log 8 - \frac{8}{x}\right) = \frac{8^x}{x^8}\left(\log 8 - \frac{8}{x}\right)$$

**Q. 27** $\log(x + \sqrt{x^2 + a})$

**Sol.** Let $$y = \log(x + \sqrt{x^2 + a})$$

$\therefore$ $$\frac{dy}{dx} = \frac{d}{dx}\log(x + \sqrt{x^2 + a})$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}.\frac{d}{dx}[x + \sqrt{x^2 + a}]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}\left[1 + \frac{1}{2}(x^2 + a)^{-1/2}\cdot 2x\right]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}\cdot\left(1 + \frac{x}{\sqrt{x^2 + a}}\right)$$

$$= \frac{(\sqrt{x^2 + a} + x)}{(x + \sqrt{x^2 + a})(\sqrt{x^2 + a})} = \frac{1}{(\sqrt{x^2 + a})}$$

**Q. 23** Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

**Sol.** We have,
$$f(x) = |x - 5|$$
$\therefore$
$$f(x) = \begin{cases} -(x-5), & \text{if } x < 5 \\ x - 5, & \text{if } x \geq 5 \end{cases}$$

For continuity at $x = 5$,
$$\text{LHL} = \lim_{x \to 5^-} (-x + 5)$$
$$= \lim_{h \to 0} [-(5 - h) + 5] = \lim_{h \to 0} h = 0$$
$$\text{RHL} = \lim_{x \to 5^+} (x - 5)$$
$$= \lim_{h \to 0} (5 + h - 5) = \lim_{h \to 0} h = 0$$
$\therefore$ $f(5) = 5 - 5 = 0$

$\Rightarrow$ $\text{LHL} = \text{RHL} = f(5)$

Hence, $f(x)$ is continuous at $x = 5$.

Now,
$$Lf'(5) = \lim_{x \to 5^-} \frac{f(x) - f(5)}{x - 5}$$
$$= \lim_{x \to 5^-} \frac{-x + 5 - 0}{x - 5} = -1$$
$$Rf'(5) = \lim_{x \to 5^+} \frac{f(x) - f(5)}{x - 5}$$
$$= \lim_{x \to 5^+} \frac{x - 5 - 0}{x - 5} = 1$$
$\therefore$ $Lf'(5) \neq Rf'(5)$

So, $f(x) = |x - 5|$ is not differentiable at $x = 5$.

**Q. 24** A function $f : R \to R$ satisfies the equation $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in R$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$, then prove that $f'(x) = 2f(x)$.

**Sol.** Let $f : R \to R$ satisfies the equation $f(x + y) = f(x) \cdot f(y)$, $\forall\ x, y \in R$, $f(x) \neq 0$.
Let $f(x)$ is differentiable at $x = 0$ and $f'(0) = 2$.

$\Rightarrow$
$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$
$\Rightarrow$
$$2 = \lim_{x \to 0} \frac{f(x) - f(0)}{x}$$
$\Rightarrow$
$$2 = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{0 + h}$$
$\Rightarrow$
$$2 = \lim_{h \to 0} \frac{f(0) \cdot f(h) - f(0)}{h}$$
$\Rightarrow$
$$2 = \lim_{h \to 0} \frac{f(0)\,[f(h) - 1]}{h} \quad [\because f(0) = f(h)] \ldots \text{(i)}$$

Also,
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h} \quad [\because f(x + y) = f(x) \cdot f(y)]$$
$$= \lim_{h \to 0} \frac{f(x)\,[f(h) - 1]}{h} = 2f(x) \quad \text{[using Eq. (i)]}$$
$\therefore$ $f'(x) = 2f(x)$

**Q. 21** $f(x)=\begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$ at $x=0$.

**Sol.** We have, $f(x)=\begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$ at $x=0$

For differentiability at $x=0$,

$$Lf'(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$

$$=\lim_{h\to 0}\frac{(0-h)^2\sin\left(\frac{1}{0-h}\right)}{0-h}=\lim_{h\to 0}\frac{h^2\sin\left(\frac{-1}{h}\right)}{-h}$$

$$=\lim_{h\to 0}+h\sin\left(\frac{1}{h}\right) \qquad [\because \sin(-\theta)=-\sin\theta]$$

$$=0\times[\text{an oscillating number between } -1 \text{and} 1]=0$$

$$Rf'(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$

$$=\lim_{h\to 0}\frac{(0+h)^2\sin\left(\frac{1}{0+h}\right)}{0+h}=\lim_{h\to 0}\frac{h^2\sin(1/h)}{h}$$

$$=\lim_{h\to 0}h\sin(1/h)$$

$$=0\times[\text{an oscillating number between } -1 \text{ and } 1]=0$$

$\because \qquad Lf'(0)=Rf'(0)$

So, $f(x)$ is differentiable at $x=0$.

**Q. 22** $f(x)=\begin{cases} 1+x, & \text{if } x\le 2 \\ 5-x, & \text{if } x>2 \end{cases}$ at $x=2$.

**Sol.** We have, $f(x)=\begin{cases} 1+x, \text{if } x\le 2 \\ 5-x, \text{if } x>2 \end{cases}$ at $x=2$.

For differentiability at $x=2$,

$$Lf'(2)=\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{(1+x)-(1+2)}{x-2}$$

$$=\lim_{h\to 0}\frac{(1+2-h)-3}{2-h-2}=\lim_{h\to 0}\frac{-h}{-h}=1$$

$$Rf'(2)=\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{(5-x)-3}{x-2}$$

$$=\lim_{h\to 0}\frac{5-(2+h)-3}{2+h-2}$$

$$=\lim_{h\to 0}\frac{5-2-h-3}{h}=\lim_{h\to 0}\frac{-h}{+h}$$

$$=-1$$

$\because \qquad Lf'(2)\neq Rf'(2)$

So, $f(x)$ is not differentiable at $x=2$.

**Q. 19** Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

**Sol.** We have, $f(x) = |\sin x + \cos x|$ at $x = \pi$

Let $g(x) = \sin x + \cos x$

and $h(x) = |x|$

$$\therefore \quad hog(x) = h[g(x)] = h(\sin x + \cos x) = |\sin x + \cos x|$$

Since, $g(x) = \sin x + \cos x$ is a continuous function as it is forming with addition of two continuous functions $\sin x$ and $\cos x$.

Also, $h(x) = |x|$ is also a continuous function. Since, we know that composite functions of two continuous functions is also a continuous function.

Hence, $f(x) = |\sin x + \cos x|$ is a continuous function everywhere.

So, $f(x)$ is continuous at $x = \pi$.

**Q. 20** Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases} \text{ at } x = 2.$$

**Thinking Process**

*We know that, a function $f$ is differentiable at a point $a$ in its domain, if both $Lf'(a)$ and $Rf'(a)$ are finite and equal, where $Lf'(c) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$ and $Rf'(c) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.*

**Sol.** We have, $f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x & \text{if } 2 \le x < 3 \end{cases}$ at $x = 2$.

At $x = 2$, $$Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{(2-h)[2-h] - (2-1)2}{-h}$$

$\{\because [a-h] = [a-1], \text{where } a \text{ is any positive number}\}$

$$= \lim_{h \to 0} \frac{(2-h)(1) - 2}{-h}$$

$$= \lim_{h \to 0} \frac{2-h-2}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

$$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h-1)(2+h) - (2-1) \cdot 2}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)(2+h) - 2}{h}$$

$$= \lim_{h \to 0} \frac{2 + h + 2h + h^2 - 2}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 3h}{h} = \lim_{h \to 0} \frac{h(h+3)}{h} = 3$$

$$\therefore \quad Lf'(2) \ne Rf'(2)$$

So, $f(x)$ is not differentiable at $x = 2$.

**Q. 17** If the function $f(x) = \frac{1}{x+2}$, then find the points of discontinuity of the composite function $y = f\{f(x)\}$.

**Sol.** We have,
$$f(x) = \frac{1}{x+2}$$
$\therefore$
$$y = f\{f(x)\}$$
$$= f\left(\frac{1}{x+2}\right) = \frac{1}{\frac{1}{x+2} + 2}$$
$$= \frac{1}{1+2x+4} \cdot (x+2) = \frac{(x+2)}{(2x+5)}$$

So, the function $y$ will not be continuous at those points, where it is not defined as it is a rational function.

Therefore, $y = \frac{x+2}{(2x+5)}$ is not defined, when $2x + 5 = 0$

$\therefore$
$$x = \frac{-5}{2}$$

Hence, $y$ is discontinuous at $x = \frac{-5}{2}$.

**Q. 18** Find all points of discontinuity of the function $f(t) = \frac{1}{t^2 + t - 2}$, where $t = \frac{1}{x-1}$.

**Sol.** We have,
$$f(t) = \frac{1}{t^2 + t - 2} \text{ and } t = \frac{1}{x-1}$$
$\therefore$
$$f(t) = \frac{1}{\left(\frac{1}{x^2+1-2x}\right) + \left(\frac{1}{x-1}\right) - \frac{2}{1}}$$
$$= \frac{1}{\left(\frac{1 + x - 1 + [-2(x-1)^2]}{(x^2+1-2x)}\right)}$$
$$= \frac{x^2+1-2x}{x - 2x^2 - 2 + 4x}$$
$$= \frac{x^2+1-2x}{-2x^2+5x-2}$$
$$= \frac{(x-1)^2}{-(2x^2-5x+2)}$$
$$= \frac{(x-1)^2}{(2x-1)(2-x)}$$

So, $f(t)$ is discontinuous at $2x - 1 = 0 \Rightarrow x = 1/2$

and $2 - x = 0 \Rightarrow x = 2$.

**Q. 15** Prove that the function $f$ defined by $f(x)=\begin{cases}\dfrac{x}{|x|+2x^2}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$

remains discontinuous at $x=0$, regardless the choice of $k$.

**Sol.** We have, $$f(x)=\begin{cases}\dfrac{x}{|x|+2x^2}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

At $x=0$, $$\text{LHL}=\lim_{x\to 0^-}\frac{x}{|x|+2x^2}=\lim_{h\to 0}\frac{(0-h)}{|0-h|+2(0-h)^2}$$
$$=\lim_{h\to 0}\frac{-h}{h+2h^2}=\lim_{h\to 0}\frac{-h}{h(1+2h)}=-1$$
$$\text{RHL}=\lim_{x\to 0^+}\frac{x}{|x|+2x^2}=\lim_{h\to 0}\frac{0+h}{|0+h|+2(0+h)^2}$$
$$=\lim_{h\to 0}\frac{h}{h+2h^2}=\lim_{h\to 0}\frac{h}{h(1+2h)}=1$$

and $f(0)=k$

Since, LHL $\neq$ RHL for any value of $k$.

Hence, $f(x)$ is discontinuous at $x=0$ regardless the choice of $k$.

**Q. 16** Find the values of $a$ and $b$ such that the function $f$ defined by
$$f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$$

is a continuous function at $x=4$.

**Sol.** We have, $$f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$$

At $x=4$, $$\text{LHL}=\lim_{x\to 4^-}\frac{x-4}{|x-4|}+a$$
$$=\lim_{h\to 0}\frac{4-h-4}{|4-h-4|}+a=\lim_{h\to 0}\frac{-h}{h}+a$$
$$=-1+a$$
$$\text{RHL}=\lim_{x\to 4^+}\frac{x-4}{|x-4|}+b$$
$$=\lim_{h\to 0}\frac{4+h-4}{|4+h-4|}+b=\lim_{h\to 0}\frac{h}{h}+b=1+b$$
$$f(4)=a+b\Rightarrow -1+a=1+b=a+b$$
$\Rightarrow$ $$-1+a=a+b \text{ and } 1+b=a+b$$
$\therefore$ $$b=-1 \text{ and } a=1$$

$\therefore$ $$\text{LHL} = \lim_{x \to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$$

$$= \lim_{x \to 0^-} \left(\frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}\right) \cdot \left(\frac{\sqrt{1+kx} + \sqrt{1-kx}}{\sqrt{1+kx} + \sqrt{1-kx}}\right)$$

$$= \lim_{x \to 0^-} \frac{1+kx-1+kx}{x[\sqrt{1+kx} + \sqrt{1-kx}]}$$

$$= \lim_{x \to 0^-} \frac{2kx}{x\sqrt{1+kx} + \sqrt{1-kx}}$$

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1+k(0-h)} + \sqrt{1-k(0-h)}}$$

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1-kh} + \sqrt{1+kh}} \qquad = \frac{2k}{2} = k$$

and $$f(0) = \frac{2 \times 0 + 1}{0 - 1} = -1$$

$\Rightarrow$ $k = -1$ $\qquad [\because \text{LHL} = \text{RHL} = f(0)]$

**Q. 14** $f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$.

**Sol.** We have, $f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$

At $x = 0$, $$\text{LHL} = \lim_{x \to 0^-} \frac{1-\cos kx}{x \sin x} = \lim_{h \to 0} \frac{1-\cos k(0-h)}{(0-h)\sin(0-h)}$$

$$= \lim_{h \to 0} \frac{1-\cos(-kh)}{-h\sin(-h)}$$

$$= \lim_{h \to 0} \frac{1-\cos kh}{h \sin h} \qquad [\because \cos(-\theta) = \cos\theta,\ \sin(-\theta) = -\sin\theta]$$

$$= \lim_{h \to 0} \frac{1-1+2\sin^2 \frac{kh}{2}}{h \sin h} \qquad \left[\because \cos\theta = 1 - 2\sin^2\frac{\theta}{2}\right]$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{kh}{2}}{h \sin h}$$

$$= \lim_{h \to 0} \frac{2\sin\frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{\sin\frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{1}{\frac{\sin h}{h}} \cdot \frac{k^2 h/4}{h}$$

$$= \frac{2k^2}{4} = \frac{k^2}{2} \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Also, $$f(0) = \frac{1}{2} \Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1 \qquad \text{p}$$

**Q. 11** $f(x)=\begin{cases}3x-8, & \text{if } x\le 5\\ 2k, & \text{if } x>5\end{cases}$ at $x=5$.

**Sol.** We have, $f(x)=\begin{cases}3x-8, & \text{if } x\le 5\\ 2k, & \text{if } x>5\end{cases}$ at $x=5$

Since, $f(x)$ is continuous at $x=5$.

$\therefore$ LHL = RHL = $f(5)$

Now, $\text{LHL}=\lim_{x\to 5^-}(3x-8)=\lim_{h\to 0}[3(5-h)-8]$

$=\lim_{h\to 0}[15-3h-8]=7$

$\text{RHL}=\lim_{x\to 5^+}2k=\lim_{h\to 0}2k=2k=7$ $\quad[\because \text{LHL}=\text{RHL}]$

and $f(5)=3\times 5-8=7$

$\therefore$ $2k=7 \Rightarrow k=\frac{7}{2}$

**Q. 12** $f(x)=\begin{cases}\dfrac{2^{x+2}-16}{4^x-16}, & \text{if } x\ne 2\\ k, & \text{if } x=2\end{cases}$ at $x=2$.

**Sol.** We have, $f(x)=\begin{cases}\dfrac{2^{x+2}-16}{4^x-16}, & \text{if } x\ne 2\\ k, & \text{if } x=2\end{cases}$ at $x=2$

Since, $f(x)$ is continuous at $x=2$.

$\therefore$ LHL = RHL = $f(2)$

At $x=2$, $\lim_{x\to 2}\dfrac{2^x\cdot 2^2-2^4}{4^x-4^2}=\lim_{x\to 2}\dfrac{4\cdot(2^x-4)}{(2^x)^2-(4)^2}$

$=\lim_{x\to 2}\dfrac{4\cdot(2^x-4)}{(2^x-4)(2^x+4)}$ $\quad[\because a^2-b^2=(a+b)(a-b)]$

$=\lim_{x\to 2}\dfrac{4}{2^x+4}=\dfrac{4}{8}=\dfrac{1}{2}$

But $f(2)=k$

$\therefore$ $k=\dfrac{1}{2}$

**Q.13** $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1\end{cases}$ at $x=0$.

**Sol.** We have, $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1\end{cases}$ at $x=0$.

$$= \lim_{h \to 0} \frac{1}{e^{-1/h} + 1} = \frac{1}{e^{-\infty} + 1}$$

$$= \frac{1}{0 + 1} = 1 \qquad [\because e^{-\infty} = 0]$$

Hence, LHL $\neq$ RHL at $x = 0$.

So, $f(x)$ is discontinuous at $x = 0$.

## Q. 9 $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ at $x = 1$.

**Sol.** We have, $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ at $x = 1$

At $x = 1$,

$$\text{HL} = \lim_{x \to 1^-} \frac{x^2}{2} = \lim_{h \to 0} \frac{(1-h)^2}{2}$$

$$= \lim_{h \to 0} \frac{1 + h^2 - 2h}{2} = \frac{1}{2}$$

$$\text{RHL} = \lim_{x \to 1^+} \left(2x^2 - 3x + \frac{3}{2}\right)$$

$$= \lim_{h \to 0} \left[2(1+h)^2 - 3(1+h) + \frac{3}{2}\right]$$

$$= \lim_{h \to 0} \left(2 + 2h^2 + 4h - 3 - 3h + \frac{3}{2}\right) = -1 + \frac{3}{2} = \frac{1}{2}$$

and $f(1) = \dfrac{1^2}{2} = \dfrac{1}{2}$

$\therefore$ LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

## Q. 10 $f(x) = |x| + |x - 1|$ at $x = 1$.

**Sol.** We have, $f(x) = |x| + |x - 1|$ at $x = 1$

At $x = 1$,

$$\text{LHL} = \lim_{x \to 1^-} [|x| + |x - 1|]$$

$$= \lim_{h \to 0} [|1 - h| + |1 - h - 1|] = 1 + 0 = 1$$

and

$$\text{RHL} = \lim_{x \to 1^-} [|x| + |x - 1|]$$

$$= \lim_{h \to 0} [|1 + h| + |1 + h - 1|] = 1 + 0 = 1$$

and $f(1) = |1| + |0| = 1$

$\therefore$ LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

**Note** *Every modulus function is a continuous function at any real point.*

**Q. 7** $f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a}, & \text{if } x \neq 0 \\ 0, & \text{if } x = a \end{cases}$ at $x = a$.

**Sol.** We have, $$f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a}, & \text{if } x \neq 0 \\ 0, & \text{if } x = a \end{cases} \text{ at } x = a$$

At $x = a$, $$\text{LHL} = \lim_{x \to a^-} |x-a| \sin \frac{1}{x-a}$$

$$= \lim_{h \to 0} |a-h-a| \sin \left( \frac{1}{a-h-a} \right)$$

$$= \lim_{h \to 0} -h \sin \left( \frac{1}{h} \right) \qquad [\because \sin(-\theta) = -\sin\theta]$$

$= 0 \times$ [an oscillating number between $-1$ and 1] $= 0$

$$\text{RHL} = \lim_{x \to a^+} |x-a| \sin \left( \frac{1}{x-a} \right)$$

$$= \lim_{h \to 0} |a+h-a| \sin \left( \frac{1}{a+h-a} \right) = \lim_{h \to 0} h \sin \frac{1}{h}$$

$= 0 \times$ [an oscillating number between $-1$ and 1] $= 0$

and $f(a) = 0$

$\therefore$ LHL = RHL = $f(a)$

So, $f(x)$ is continuous at $x = a$.

**Q. 8** $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$.

**Sol.** We have, $$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ at } x = 0$$

At $x = 0$, $$\text{LHL} = \lim_{x \to 0^-} \frac{e^{1/x}}{1+e^{1/x}} = \lim_{h \to 0} \frac{e^{1/0-h}}{1+e^{1/0-h}}$$

$$= \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{-1/h}} = \lim_{h \to 0} \frac{1}{e^{1/h}(1+e^{-1/h})}$$

$$= \lim_{h \to 0} \frac{1}{e^{1/h}+1} = \frac{1}{e^{\infty}+1} = \frac{1}{\infty+1} \qquad [\because e^{\infty} = \infty]$$

$$= \frac{1}{\frac{1}{0}} = 0$$

$$\text{RHL} = \lim_{x \to 0^+} \frac{e^{1/x}}{1+e^{1/x}}$$

$$= \lim_{h \to 0} \frac{e^{1/0+h}}{1+e^{1/0+h}} = \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}}$$

$$= \lim_{h \to 0} \frac{2(2+h)^2 - 3(2+h) - 2}{(2+h) - 2}$$

$$= \lim_{h \to 0} \frac{8 + 2h^2 + 8h - 6 - 3h - 2}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 5h}{h} = \lim_{h \to 0} \frac{h(2h+5)}{h} = 5$$

and $\quad f(2) = 5$

$\therefore \quad \text{LHL} = \text{RHL} = f(2)$

So, $f(x)$ is continuous at $x = 2$.

**Q. 5** $f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ **at** $x = 4$.

**Sol.** We have, $\quad f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ at $x = 4$.

At $x = 4$, $\quad \text{LHL} = \lim_{x \to 4^-} \dfrac{|x-4|}{2(x-4)}$

$$= \lim_{h \to 0} \frac{|4-h-4|}{2[(4-h)-4]} = \lim_{h \to 0} \frac{|0-h|}{(8-2h-8)}$$

$$= \lim_{h \to 0} \frac{h}{-2h} = \frac{-1}{2} \quad \text{and} \quad f(4) = 0 \neq \text{LHL}$$

So, $f(x)$ is discontinuous at $x = 4$.

**Q. 6** $f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ **at** $x = 0$.

**Sol.** We have, $\quad f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

At $x = 0$, $\quad \text{LHL} = \lim_{x \to 0^-} |x| \cos \dfrac{1}{x} = \lim_{h \to 0} |0-h| \cos \dfrac{1}{0-h}$

$$= \lim_{h \to 0} h \cos \left(\frac{-1}{h}\right)$$

$= 0 \times$[an oscillating number between $-1$ and $1$] $= 0$

$$\text{RHL} = \lim_{x \to 0^+} |x| \cos \frac{1}{x}$$

$$= \lim_{h \to 0} |0+h| \cos \frac{1}{(0+h)}$$

$$= \lim_{h \to 0} h \cos \frac{1}{h}$$

$= 0 \times$[an oscillating number between $-1$ and $1$] $= 0$

and $\quad f(0) = 0$

Since, $\quad \text{LHL} = \text{RHL} = f(0)$

So, $f(x)$ is continuous at $x = 0$.

**Q. 3** $f(x)=\begin{cases}\dfrac{1-\cos 2x}{x^2}, & \text{if } x\neq 0\\ 5, & \text{if } x=0\end{cases}$ at $x=0$.

**Sol.** We have, $f(x)=\begin{cases}\dfrac{1-\cos 2x}{x^2}, & \text{if } x\neq 0\\ 5, & \text{if } x=0\end{cases}$ at $x=0$.

At $x=0$,

$$\text{LHL}=\lim_{x\to 0^-}\frac{1-\cos 2x}{x^2}$$

$$=\lim_{h\to 0}\frac{1-\cos 2(0-h)}{(0-h)^2}$$

$$=\lim_{h\to 0}\frac{1-\cos 2h}{h^2} \qquad [\because \cos(-\theta)=\cos\theta]$$

$$=\lim_{h\to 0}\frac{1-1+2\sin^2 h}{h^2} \qquad [\because \cos 2\theta=1-2\sin^2\theta]$$

$$=\lim_{h\to 0}\frac{2(\sin h)^2}{(h)^2} \qquad \left[\because \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$=2$$

$$\text{RHL}=\lim_{x\to 0^+}\frac{1-\cos 2x}{x^2}$$

$$=\lim_{h\to 0}\frac{1-\cos 2(0+h)}{(0+h)^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 h}{h^2}=2 \qquad \left[\because \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

and $f(0)=5$

Since, $\text{LHL}=\text{RHL}\neq f(0)$

Hence, $f(x)$ is not continuous at $x=0$.

**Q. 4** $f(x)=\begin{cases}\dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2\\ 5, & \text{if } x=2\end{cases}$ at $x=2$.

**Sol.** We have, $f(x)=\begin{cases}\dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2\\ 5, & \text{if } x=2\end{cases}$ at $x=2$.

At $x=2$,

$$\text{LHL}=\lim_{x\to 2^-}\frac{2x^2-3x-2}{x-2}$$

$$=\lim_{h\to 0}\frac{2(2-h)^2-3(2-h)-2}{(2-h)-2}$$

$$=\lim_{h\to 0}\frac{8+2h^2-8h-6+3h-2}{-h}$$

$$=\lim_{h\to 0}\frac{2h^2-5h}{-h}=\lim_{h\to 0}\frac{h(2h-5)}{-h}=5$$

$$\text{RHL}=\lim_{x\to 2^+}\frac{2x^2-3x-2}{x-2}$$

# 5

# Continuity and Differentiability

## Short Answer Type Questions

**Q. 1** Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

**Thinking Process**

*We know that, function $f$ will be continuous at $x = a$, if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.*

**Sol.** We have, $$f(x) = x^3 + 2x^2 - 1 \text{ at } x = 1.$$

$\therefore$ $$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} (1 + h)^3 + 2(1 + h)^2 - 1 = 2$$

and $$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} (1 - h)^3 + 2(1 - h)^2 - 1 = 2$$

$\therefore$ $$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$

and $$f(1) = 1 + 2 - 1 = 2$$

So, $f(x)$ is continuous at $x = 1$.

***Note*** *Every polynomial function is continuous at any real point.*

**Q. 2** $f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$ at $x = 2$.

**Sol.** We have, $$f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases} \text{ at } x = 2.$$

At $x = 2$, $$\text{LHL} = \lim_{x \to 2^-} (x)^2$$
$$= \lim_{h \to 0} (2 - h)^2 = \lim_{h \to 0} (4 + h^2 - 4h) = 4$$

and $$\text{RHL} = \lim_{x \to 2^+} (3x + 5)$$
$$= \lim_{h \to 0} [3(2 + h) + 5] = 11$$

Since, $$\text{LHL} \neq \text{RHL} \text{ at } x = 2$$

So, $f(x)$ is discontinuous at $x = 2$.